Standing Waves in a Sphere

Given a sphere composed of an elastic media, sound waves confined to its interior or to its surface exhibit unique characteristics. The fundamental frequencies will be determined along with the general equation for these two types of sound waves for an initial disturbance at a point in or on the sphere. The results should reasonably model the sound made by a basketball after it bounces.

This work is incomplete and unreviewed. It may therefore contain inaccuracies and should not be used for educational purposes. As long as I am actively thinking about this, I am hopeful that I can find and correct any mistakes. However, please also note that I am not striving for absolute mathematical rigor here. Because this is recreational math, I am content when I have found a solution that I can verify numerically and experimentally. I strongly believe that convergence issues are vitally important, but in a more serious context than this. Please accept this material in that light.

Larry Smith, March, 1998.

Internal Sound Waves

We assume that sound in the interior of a sphere of radius d is governed by a wave equation

$$\nabla^2 u = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

where u is a function of a spatial coordinate $\mathbf{x}$ and time t. The constant c is the speed of the sound wave in the interior of the sphere. (It is also possible to introduce a dispersion effect by adding a first derivative of u with respect to t.)

Physically, for sound waves in a fluid at any rate, u represents the deviation of the pressure from the average. In this interpretation, $c = (B/\rho_0)^{1/2}$ where B is the bulk modulus, γp_0 and ρ_0 is the density of the fluid. In the derivation of the wave equation, it is also assumed that

$$\frac{\partial \mathbf{v}}{\partial t} = -\frac{1}{\rho_0}\nabla u$$

where $\mathbf{v}$ is the velocity of the fluid. For a sound wave confined to a container, the velocity $\mathbf{v}$ must be perpendicular to the surface of the container.

Applying separation of variables, let $u(\mathbf{x}, t) = A(\mathbf{x})W(t)$ to obtain

$$\frac{1}{A}\nabla^2 A = \frac{1}{c^2 W}\frac{d^2 W}{dt^2}.$$

Since the two sides depend on disjoint variables, they must equal a constant whose physical dimension is inverse square distance, denote this by $-k^2 = -1/\lambda^2$. The sign of the constant will be justified below, but for now there is no loss of generality as k could be imaginary.

The equation for W is then

$$\frac{d^2 W}{dt^2} + (kc)^2 W = 0$$

which has solutions $\sin(kct)$ and $\cos(kct)$. For our purposes, since the initial disturbance is at a point, the solution will be time symmetric and hence we need only consider the cos solution. If a first derivative appeared, then the general solution of the second order ordinary differential equation would exhibit decaying exponentials. This could prove to be useful in predicting the rapidity with which the lower fundamental frequencies dominate the sound waves in the sphere.

The equation for A is

$$\nabla^2 A + k^2 A = 0$$

which is known as Helmholtz's equation. We now turn to solving this equation by separation of variables in spherical coordinates.

To be explicit about the coordinates, fix a right-handed Euclidean coordinate system. For a point P, let r denote the distance from the origin to P, let ϕ denote the angle between the z-axis and the ray from the origin to P, and let θ denote the angle measured from the x-axis to the ray from the origin to the point

P' which is the perpendicular projection of P onto the xy-plane. The angle θ is measured positively from the positive x-axis towards the positive y-axis.

The expression for Helmholtz's equation in spherical coordinates is

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial A}{\partial r}) + \frac{1}{r^2\sin^2\phi}\frac{\partial^2 A}{\partial \theta^2} + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial \phi}(\sin\phi\frac{\partial A}{\partial \phi}) + k^2A = 0.$$

Since the solutions we seek have an initial distribution at a point, we may assume that it lies on the z-axis. Consequently, the solution will remain independent of θ for all time. So let $A(r,\theta,\phi) = R(r)\Phi(\phi)$. Separating the variables we get

$$\frac{1}{R}\frac{d}{dr}(r^2\frac{dR}{dr}) + k^2r^2 = -\frac{1}{\Phi}\frac{1}{\sin\ \phi}\frac{d}{d\phi}(\sin\ \phi\frac{d\Phi}{d\phi}).$$

Each side must equal a constant, call this β for now.

The equation for Φ becomes

$$\frac{1}{\sin\ \phi}\frac{d}{d\phi}(\sin\ \phi\frac{d\Phi}{d\phi}) + \beta\Phi = 0,$$

which is Legendre's equation. It is well known that the only continuous solutions to this equation on the interval $[0,\pi]$ occur when $\beta = n(n+1)$ where $n = 0, 1, \ldots$ and the solutions are $P_n(\cos\ \phi)$ where P_n is the nth Legendre polynomial. The system $P_n(\cos\ \phi)$ is orthogonal with respect to integration by $\sin\ \phi\ d(\phi)$, that is

$$||P_n||^2 = \int_0^\pi P_n^2(\cos\ \phi)\sin\ \phi d\phi = \int_{-1}^1 P_n^2(x)dx = \frac{2}{2n+1}.$$

We now know that $\beta = n(n+1)$ for some integer n, hence we can write the resulting equation for R as

$$\frac{d}{dr}(r^2\frac{dR}{dr}) + (k^2r^2 - n(n+1))R = 0.$$

The boundary conditions are found at $r = 0$ simply by continuity, namely $R'(0) = 0$, for if $R'(0) \neq 0$ then the derivative of A will be different in different directions. At $r = d$, the boundary condition is the same for all sound waves in a container, namely the normal component of the velocity is 0. Since the time derivative of the velocity is proportional to the gradient of the pressure, we require that the pressure be normal to the surface. And since the surface normal is in the radial direction, the requirement is that $R'(d) = 0$.

We now have a Sturm-Liouville problem for R, so there are a countable infinity of solutions, defining the parameter k, and the solutions are orthogonal on $[0,d]$ with respect to weighting by r^2. It is well known that the only continuous solutions to the equation are the spherical Bessel functions $R(r) = j_n(kr)$ with k real. Since $R'(r) = kj_n'(kr)$, we require $j_n'(0) = 0$ which is automatic (check this) and $j_n'(kd) = 0$. For a fixed n let $k_{n,m}$ be the mth positive root of the equation $j_n'(kd) = 0$. Then the $k_{n,m}$ are the eigenvalues of the equation with eigenfunctions $j_n(k_{n,m}r)$.

For n fixed, the collection $j_n(k_{n,m}r)$ is orthogonal. The norm of these functions will also be required. By definition,

$$j_n(x) = (\pi/2x)^{1/2}J_{n+1/2}(x).$$

Substituting $x = kr$, square, multiply by r^2, and then integrate to get the norm

$$\begin{aligned} ||j_n(kr)||^2 &= \frac{\pi}{2k}\int_0^d rJ_{n+1/2}^2(kr)dr \\ &= \frac{\pi}{2k}||J_{n+1/2}(kr)||^2. \end{aligned}$$

Where $J_{n+1/2}$ is the Bessel function of the first kind and the weighting for the norm is r. In order to apply the formula for the norm of J, we must exhibit the boundary conditions that it satisfies. From the definition of j,

$$j_n'(x) = (\pi/2)^{1/2}(-\frac{1}{2}x^{-3/2}J_{n+1/2}(x) + x^{-1/2}J_{n+1/2}'(x)).$$

Set this equal to 0, and $x = kd$, and multiply by constants to find that k satisfies

$$\frac{-1}{2} J_{n+1/2}(kd) + (kd)J'_{n+1/2}(kd) = 0.$$

This is of the required form

$$hJ_{n+1/2}(kd) + (kd)J'_{n+1/2}(kd) = 0,$$

where $h = -1/2$ for which it is known that the norm is

$$||J_{n+1/2}(kr)||^2 = \frac{k^2d^2 - (n+1/2)^2 + h^2}{2k^2}[J_{n+1/2}(kd)]^2.$$

Rearranging,

$$||j_n(kr)||^2 = d^3 \frac{(kd)^2 - n(n+1)}{2(kd)^2}[j_n(kd)]^2.$$

We can now write down the most general form of a sound wave inside a sphere. Let $k_{n,m}$ as above, then the general solution is

$$\sum_{n=0,m=1}^{\infty} a_{n,m}\ j_n(k_{n,m}r)P_n(\cos\ \phi)\cos(k_{n,m}ct)$$

for suitable real constants $a_{n,m}$.

In general, for an initial disturbance at time 0 given by some $f(r,\phi)$, we can solve for $a_{n,m}$ by setting $t = 0$ in the series solution and equating to $f(r,\phi)$. Multiply both sides by

$$j_p(k_{p,q}r)P_p(\cos\ \phi)r^2 \sin\ \phi d\theta d\phi dr,$$

and integrate over the sphere. The right hand integral over $f(r,\phi)$ evaluates numerically for each p,q

$$f_{p,q} = \int_0^d \int_0^\pi \int_0^{2\pi} f(r,\phi)j_p(k_{p,q}r)P_p(\cos\ \phi)r^2 \sin\ \phi d\theta d\phi dr.$$

Term by term integration of the left hand side leaves

$$\sum_{n=0,m=1}^{\infty} a_{n,m} \int_0^d r^2 j_n(k_{n,m}r)j_p(k_{p,q}r)dr \int_0^\pi P_n(\cos\ \phi)P_p(\cos\ \phi)d(\cos\ \phi) \int_0^{2\pi} d\theta.$$

By orthogonality, all of the terms except where $n = p$ and $m = q$ are 0. Consequently, $a_{p,q}$ are found from

$$2\pi a_{p,q}||j_p(k_{p,q}r)||^2||P_n||^2 = f_{p,q}.$$

Internal Waves for an Initial Disturbance at $z = d$

We will now consider a point disturbance on the z-axis. A correct physical model for this disturbance might, for example, impart an instantaneous impulse to the fluid at a point. This approach may or may not be tractable, but the mathematical difficulty that would arise is not unlike the much simpler model in which a prescribed value for u is given at $t = 0$.

So let us assume that the initial disturbance is on the positive z-axis a distance $z < d$ from the origin at time $t = 0$. It is represented by a delta function δ at the point $r = z, \phi = 0$. Anyone who has followed a graduate level course in analysis knows that the Dirac delta function is an abomination of mathematics. The correct notion is that of signed measure, and wherever we want to use a Dirac delta function, we must interpret functions as kernels of signed measures. Convergence of series is then seen as convergence of signed measures, etc. With that understood, I am going to use the Dirac delta notation in hopes that this text will be accessible to undergraduates in physics.

Now the $f_{p,q}$ are found easily because the Dirac delta function evaluates the integrand at the point of the disturbance, ignoring the volume element. Thus $f_{p,q} = j_p(k_{p,q}z)$ which has used the fact that $P_p(1) = 1$. Therefore,

$$2\pi a_{p,q}||j_p(k_{p,q}r)||^2||P_n||^2 = j_p(k_{p,q}z).$$

Substituting the expressions for the norms,

$$2\pi a_{p,q}d^3\frac{(k_{p,q}d)^2 - p(p+1)}{2(k_{p,q}d)^2}[j_p(k_{p,q}d)]^2\frac{2}{2p+1} = j_p(k_{p,q}z).$$

Solving for $a_{p,q}$, and taking the limit as $z \to d$,

$$d^3a_{p,q} = \frac{1}{\pi}\frac{(p+1/2)(k_{p,q}d)^2}{j_p(k_{p,q}d)((k_{p,q}d)^2 - p(p+1))}.$$

The quantities $k_{p,q}d$ are the zeros of $j_p'(x)$ so explicit values for $d^3a_{p,q}$ valid for any d can be tabulated for experimental verification.

This series is not easily summable by numeric methods. To aid in the visualization of the solution, and in the numerical calculation, a kind of asymptotic expansion will be considered. It is known that for large n and large x, that

$$j_n(x) \sim \frac{1}{x}\cos(x - \frac{n\pi}{2} - \frac{\pi}{2}).$$

For $n = 2k$ this is

$$j_{2k}(x) \sim (-1)^{k-1}\frac{\sin\ x}{x}$$

and for $n = 2k-1$,

$$j_{2k-1}(x) \sim (-1)^k\frac{\cos\ x}{x}.$$

For even n, the zeros of j_n' in the asymptotic expansion occur where $x = \tan\ x$. For large m therefore the values of $k_{n,m}$ approach $k_{n,m}d = (m-1/2)\pi$. By a similar calculation, if n is odd then the zeros of j_n' occur near $x = -\cot\ x$, so for large m, $k_{n,m}d = m - 1\pi$.

When $n = 2k$ is even, the value of j_n at $k_{n,m}d$ is found by

$$\begin{aligned} j_{2k}(k_{n,m}d) &= (-1)^{k-1}\sin\ k_{n,m}d/k_{n,m}d \\ &= (-1)^{k-1}\cos\ k_{n,m}d \\ &= (-1)^{k-1}(-1)^{m+1}((k_{n,m}d)^2+1)^{-1/2}. \end{aligned}$$

Here the expression for the absolute value of cos is found from the expression $\tan\ k_{n,m}d = k_{n,m}d$ and the sign of the cosine is estimated by considering the location of the successive solutions to $\tan\ x = x$.

When $n = 2k-1$ is odd, the value of j_n at $k_{n,m}d$ is found by

$$\begin{aligned} j_{2k-1}(k_{n,m}d) &= (-1)^k\cos\ k_{n,m}d/k_{n,m}d \\ &= (-1)^k\sin\ k_{n,m}d \\ &= (-1)^k(-1)^{m+1}((k_{n,m}d)^2+1)^{-1/2}. \end{aligned}$$

Here the expression for the absolute value of cos is found from the expression $\cot\ k_{n,m}d = -k_{n,m}d$ and the sign of the sine is estimated by considering the location of the successive solutions to $\cot\ x = -x$.

Since the expressions are the same in both cases (with $n = 2k$ or $n = 2k-1$), in the limit as $m \to \infty$ we find that

$$d^2a_{n,m} \sim \pm\frac{(-1)^{k-1}(-1)^{m+1}}{\pi}(n+1/2)k_{n,m}d.$$

Now suppose we use this limiting asymptotic approximation to evaluate the summation. The even n terms of the series are

$$\sum_{m=1}^{\infty} a_{n,m} j_n(k_{n,m} r) P_n(\cos\ \phi) \cos(k_{n,m} ct)$$

$$= \sum_{m=1} \frac{(-1)^{k-1}(-1)^{m+1}}{\pi}(n+1/2)\frac{k_{n,m}}{d}(-1)^{k-1}\frac{\sin\ (k_{n,m} r)}{k_{n,m} r} P_n(\cos\ \phi)\cos(k_{n,m} ct)$$

$$= \frac{1}{\pi d r} \sum_{m=1} (-1)^{m+1} \sin\ (k_{n,m} r) \cos(k_{n,m} ct)(n+1/2) P_n(\cos\ \phi)$$

$$= \frac{1}{2\pi d r}\big(\sum_{m=1} (-1)^{m+1} \sin\ k_{n,m}(r+ct) - (-1)^{m+1} \sin\ k_{n,m}(r-ct)\big)(n+1/2)P_n(\cos\ \phi).$$

$$= \frac{1}{2\pi d r}\big(\sum_{m=1} (-1)^{m+1} \sin\ (m-1/2)\pi(r+ct)$$

$$- (-1)^{m+1} \sin\ (m-1/2)\pi(r-ct)/d\big)(n+1/2)P_n(\cos\ \phi).$$

For odd n, sin is replaced by cos.

$$\sum_{m=1}^{\infty} a_{n,m} j_n(k_{n,m} r) P_n(\cos\ \phi) \cos(k_{n,m} ct)$$

$$= \sum_{m=1} \frac{(-1)^{k}(-1)^{m+1}}{\pi}(n+1/2)\frac{k_{n,m}}{d}(-1)^{k}\frac{\cos\ (k_{n,m} r)}{k_{n,m} r} P_n(\cos\ \phi)\cos(k_{n,m} ct)$$

$$= \frac{1}{\pi d r} \sum_{m=1} (-1)^{m+1} \cos\ (k_{n,m} r) \cos(k_{n,m} ct)(n+1/2) P_n(\cos\ \phi)$$

$$= \frac{1}{2\pi d r}\big(\sum_{m=1} (-1)^{m+1} \cos\ k_{n,m}(r+ct) + (-1)^{m+1} \cos\ k_{n,m}(r-ct)\big)(n+1/2)P_n(\cos\ \phi).$$

$$= \frac{1}{2\pi d r}\big(\sum_{m=1} (-1)^{m+1} \cos\ m\pi(r+ct)/d + (-1)^{m+1} \cos\ m\pi(r-ct)/d\big)(n+1/2)P_n(\cos\ \phi).$$

We now need to interpret the sums of the form

$$f(x) \sim \sum_{m=1}^{\infty} (-1)^{m+1} \sin\ (m-1/2)\pi x$$

and

$$g(x) \sim \sum_{m=1}^{\infty} (-1)^{m+1} \cos\ m\pi x.$$

It is most straightforward to identify these series by looking at their integrals, which is another way of considering weak convergence of signed measures alluded to above.

The term by term integration of $f(x)$ gives the series

$$F(x) = \sum_{m=1}^{\infty} (-1)^{m+1} \frac{\cos\ (m-1/2)\pi x}{(m-1/2)\pi} = \frac{2}{\pi}\sum_{m=1}^{\infty} (-1)^{m+1} \frac{\cos((2m-1)\frac{\pi x}{2})}{2m-1}.$$

Recall that the function that is equal to 1 for $-\pi/2 < x < \pi/2$ and -1 for $x > \pi/2$ has Fourier cosine series

$$\frac{4}{\pi}\sum_{m=1}^{\infty} (-1)^{m+1} \frac{\cos(2m-1)x}{2m-1}$$

on the range $(-\pi, \pi)$. So the series for $F(x)$ can be recognized as the function equal to $-1/2$ for $|x| < 1$, $1/2$ for $1 < |x| < 2$, $F(1) = 0$, and the pattern repeats on intervals $(-2, 2]$. From this we infer the series for $F(x)$ represents the signed measure

$$F(x) = \begin{cases} 0 & \text{for } x \text{ an odd integer} \\ -1/2 & \text{if } x \text{ is between } (4k-1, 4k+1) \text{ with } k \text{ an integer, and} \\ 1/2 & \text{if } x \text{ is between } (4k+1, 4k+3). \end{cases}$$

Therefore $f(x)$ is represented by a -1 Dirac delta function at points $4k-1$ and a $+1$ Dirac delta function at points $4k+1$ for every integer k.

By the same token, term by term integration of $g(x)$ gives the series

$$\sum_{m=1}^{\infty} (-1)^m \frac{\sin\ m\pi x}{m\pi} = \frac{-1}{\pi} \sum_{m=1}^{\infty} (-1)^{m+1} \frac{\sin\ m\pi x}{m}.$$

Again recall that the function x has Fourier sine series on $(-\pi, \pi)$ given by

$$2 \sum_{m=1}^{\infty} (-1)^{m+1} \frac{\sin\ mx}{m}.$$

And from this we infer that the series for $G(x)$ represents the signed measure

$$G(x) = \begin{cases} 2k - \frac{1}{2}x & \text{for } |2k - x| < 1 \text{ with } k \text{ an integer,} \\ 0 & \text{if } x \text{ is an odd integer.} \end{cases}$$

Therefore $g(x)$ is represented by $-1/2$ and $+1$ Dirac delta functions at the odd integers.

With these functions we can write the even terms as

$$\frac{1}{2\pi dr}(f(\frac{r+ct}{d}) - f(\frac{r-ct}{d}))(n+1/2)P_n(\cos\ \phi)$$

and the odd terms as

$$\frac{1}{2\pi dr}(g(\frac{r+ct}{d}) + g(\frac{r-ct}{d}))(n+1/2)P_n(\cos\ \phi)$$

We now need to sum $(n+1/2)P_n(\cos\ \phi)$ over even and odd n, denote these by τ_e and τ_o respectively, and find what they represent. Again, we integrate with $x = \cos\ \phi$ from $x = 0$ to $x = y$, and we use the recurrence relation

$$(2n+1)P_n(x) = P'_{n+1}(x) - P'_{n-1}(x).$$

Integrating the series term by term therefore results in a telescoping sum. The partial sum to p in the even summation reduces to

$$y + (P_{p+1}(y) - P_{p+1}(0)) - (P_1 y - P_1(0)),$$

where the integral over P_0 is done directly. Recall that $P_1(y) = y$. So in the limit the summation for $-1 < y < 1$ is -1. For $y = 1$, the limit is 0. With $y = -1$, since $p+1$ is odd the limit is -2. Summarizing, τ_e represents a one-sided Dirac delta function at 1 and -1. The partial sum to p in the odd summation reduces to

$$(P_{p+1}(y) - P_{p+1}(0)) - (P_0 y - P_0(0)).$$

Recall that $P_0(y) = 1$. For $-1 < y < 1$, the limit of $P_{p+1}(y)$ as $p \to \infty$ is 0. Therefore the summation is 0 for $-1 < y < 1$. For $y = 1$ the summation is 1. For $y = -1$ since $p+1$ is odd, the summation is -1. Thus τ_o represents a one-sided Dirac delta function at 1 and a negative one-sided Dirac delta function at -1.

We now can give a picture of this asymptotic expression for the wave inside of the sphere. It is concentrated on the z-axis (this is $\phi = 0$ and $\phi = \pi$ corresponding to the spikes of τ_o and τ_e). There is a standing -1 value on the positive z-axis and a $+1$ on the negative z-axis, and a single magnitude 2 pulse that starts at $r = d$ and travels at a constant speed c, changing sign at $r = 0$ and reversing direction at $r = -d$.

This completes the picture of the asymptotic expansion. We do not expect to actually see this solution. However, we hope now that it will be numerically feasible to compute the difference between this artificial solution and the correct series. To that end, we need to determine the error term between the asymptotic expression that was used here and the correct terms of the series.

Surface Waves

The wave equation applies to the surface,

$$\nabla^2 w = \frac{1}{s^2} \frac{\partial^2 w}{\partial t^2}$$

where w denotes the deviation of the surface in the normal direction from its equilibrium position. The speed of waves in the surface is denoted by s and its value and physical origin is fundamentally different from the speed of sound in the interior of the sphere. Attenuation can also be incorporated into this equation. The separation of variables proceeds exactly as in the case of internal waves, and we will simply recall the solution as needed.

Helmholtz's equation reduces to an ordinary differential equation since the space component of w depends only on ϕ. From above,

$$\frac{1}{d^2 \sin\ \phi} \frac{d}{d\phi} (\sin\ \phi\ \frac{d\Phi}{d\phi}) + k^2 \phi = 0.$$

As before, this is Legendre's equation so it follows that $(dk)^2 = n(n+1)$ for $n = 0, 1, \ldots$, and the corresponding solutions are $\Phi(\phi) = P_n(\cos\ \phi)$.

Therefore the general form for waves on the surface of the sphere is

$$\sum_{n=0}^{\infty} b_n P_n(\cos\ \phi) cos(k_n st)$$

where $k_n = (n(n+1))^{1/2}/d$.

If the initial displacement is given by $f(\phi)$ then we set $t = 0$ and equate the general solution to $f(\phi)$. Then we multiply both sides by the surface element

$$P_p(\cos\ \phi) d^2 \sin\ \phi d\phi\ d\theta$$

and integrate over the surface. The right hand side evaluates numerically to

$$f_p = \int_0^{2\pi} \int_0^{\pi} f(\phi) P_p(\cos\ \phi) d^2 \sin\ \phi d\phi\ d\theta$$

while the left hand side integrates term by term

$$\begin{aligned} f_p &= 2\pi d^2 \sum_{n=0}^{\infty} b_n \int_0^{\pi} P_n(\cos\ \phi) P_p(\cos\ \phi) \sin\ \phi d\phi \\ &= 2\pi d^2 b_p ||P_p||^2. \end{aligned}$$

Experimental Verification

The amplitude solutions for both the internal and surface waves are unchanged if a wave attenuation is incorporated in the original wave equation. Because of this, little additional work is required to model the decay of the sound signal. The solutions can also be used to feed into a sound generator to simulate the predicted sound, and for this only the audible frequencies need be computed.

The actual sound of a basketball can be recorded and its frequencies analyzed to compare with the predicted frequencies. The two effects of surface vibration and internal sound can be combined into a single solution, where the boundary condition for the internal vibration is that at the surface, the internal velocity equals the velocity of the oscillating surface, assuming the effect of the sound waves is minimal on the surface vibration itself. The initial condition is that at time 0 there is no pressure disturbance at all.