The Physics of Bowling

Bowling is the recreational sport in which an individual bowls a sphere along the surface of a lane attempting to knock down pins standing at the opposite end of the lane. Rules govern the size and weight of the ball, the shape and weight of the pins, the location of the pins, and the size and surface condition of the lane itself.

In this document, the physics of bowling a bowling ball is described and solved mathematically. There are two core problems in bowling that will be considered: the result of the ball colliding with the pins, and the effect of spin on the trajectory of the ball on the lane. A physiological description of the aim and release of the ball would complete the scientific study of this game.

This work is incomplete and unreviewed. It may therefore contain inaccuracies and should not be used for educational purposes.

Larry Smith, May, 1998.

Coordinate System

The coordinate system will be taken with x measured from the centerline extending positive to the right; y is measured from the center of the head pin positive towards the rear of the lane. Finally, z will be measured positive from the ground up. The directions of x, y, z coordinates will be denoted by $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.

The standard of measure will be inches, pounds, and seconds unless otherwise noted. The length of the lane from foul line to head pin is L, so the foul line is at $-L$. The radius of the ball is R and its mass is M, where $M = w/g$, w is the weight in pounds, and g is the acceleration of gravity $-32 f/s/s$. The moment of inertia of such a sphere is $I = \frac{2}{5}R^2 M$. For a non-uniform distribution of weight, the general moment of inertia will have the form $I = \beta\ R^2 M$ for some numerical constant β depending on the geometry.

The Rolling Ball

When the bowler releases the ball, the initial velocity $\mathbf{v}_0$ and angular speed ω is determined and to a great extent at the will of the bowler. These two quantities, together with the condition of the surface on which the ball is rolling, completely determine the trajectory of the ball up until the point at which it hits a pin. We seek therefore to describe the mechanics of the rolling ball, which applies generally. Before starting however, keep in mind that a ball that is bowled without any initial spin will roll in a perfectly straight line. It is conceivable that a bowler would want to bowl a straight ball, but it will be shown later exactly how a hooked ball can increase a bowler's average.

There are two kinds of forces acting on the ball: friction and drag. Friction is the force on any object that is *sliding* against another object. This force is known to be proportional to the force that is pushing the objects together and directly opposed to the direction of the slide. The constant of proportionality, which will be denoted μ, is called the *coefficient of friction*, or more accurately, the coefficient of *dynamic* friction. (*Static* friction is the force that must be overcome to start two objects sliding against each other and does not occur in this situation.) Dynamic friction occurs whenever the ball is not rolling perfectly on the surface. The other type of force is exerted on a ball that is rolling perfectly on the surface. Although there is little or no friction, since the surfaces are not sliding, a ball rolling on a surface will obviously stop eventually. However, the force slowing the ball down is not friction but is related to the imperfections of the surfaces in contact, resulting in inelastic collisions and corresponding loss of energy. This force has a dependence on both the normal force and the velocity for a given ball diameter and surface properties. One could also expect that the force of drag may even have a vertical component, since balls or tires that are rolling very fast can be observed to be *bouncing* off of an otherwise flat surface. Over the range of velocities of interest, it is possible to approximate the (horizontal) force by a constant, and so it may in fact take the same form as a frictional force. In what follows, it will be assumed that the force of friction (when it is acting) dominates the force of drag. In order to describe the motion of non-sliding balls over a long period of time it may be necessary to take the drag force into account.

Whether friction, drag, or any other conceivable force acts on a ball that is sliding on a surface, the force $\mathbf{F}$ acts on the ball at the point of contact of the ball with the surface. This force accelerates the center of mass, and at the same time generates a torque $\mathbf{F} \times\ R\mathbf{k}$ affecting the rotation of the ball. Because the ball remains on the surface, the vertical component of force does not take part in the motion of the ball. Similarly, the vertical component of the force (if any) since it is directed through the center does not change the angular velocity. Therefore we can assume that the vertical component of the force $\mathbf{F}$ is zero, and doing so will simplify the conclusion we are about to draw. From Newton's laws therefore

$$\begin{aligned} M\frac{d\mathbf{v}}{dt} &= \mathbf{F} \\ I\frac{d\omega}{dt} &= (-R\mathbf{k}) \times \ \mathbf{F}. \end{aligned}$$

Since $\mathbf{F}$ appears in both equations, it is possible to eliminate it and derive a constant of motion. Take the cross product of $R\mathbf{k}$ times the first equation and add to get

$$R\mathbf{k} \times \ M\frac{d\mathbf{v}}{dt} + I\frac{d\omega}{dt} = 0$$

Rewriting this,

$$\frac{d}{dt}(I\omega - RM\mathbf{v} \times \ \mathbf{k}) = 0,$$

which says that the vector quantity $\mathbf{C} = I\omega - RM\mathbf{v} \times \ \mathbf{k}$ remains constant throughout the motion.

If $\mathbf{v}_0$ and ω_0 are the initial velocity and angular momentum then $\mathbf{C} = I\omega_0 - RM\mathbf{v}_0 \times \ \mathbf{k}$.

We might have inferred that this quantity is conserved by noting that $\mathbf{C}$ is the angular momentum of the ball relative to the point of contact with the surface. Since the force acts through this point, it cannot change the angular momentum relative to this point, thus it remains constant.

Assume now that $\mathbf{F}$ is a frictional force. At a given instant the ball is travelling with velocity $\mathbf{v}$ along the lane (i.e. $v_z = 0$) and angular velocity ω. The vector from the center of the ball to the point of contact is $-R\mathbf{k}$, so the velocity of that part of the ball relative to the center is $\omega \times \ (-R\mathbf{k})$. Therefore, relative to the lane surface, that part of the ball is moving with velocity $\mathbf{v} - R\omega \times \ \mathbf{k}$. The frictional force F on the ball is in the direction opposed to this and proportional to the weight Mg of the ball. The proportionality constant, μ is the coefficient of friction:

$$\mathbf{F} = -\mu \ Mg\frac{\mathbf{v} - R\omega \times \ \mathbf{k}}{|\mathbf{v} - R\omega \times \ \mathbf{k}|} = -\mu \ Mg < \mathbf{v} - R\omega \times \ \mathbf{k} > .$$

Here $< \mathbf{x} >$ denotes the unit vector in the direction of $\mathbf{x}$, i.e. $< \mathbf{x} >= \mathbf{x}/|\mathbf{x}|$.

The conserved quantity $\mathbf{C}$ allows us to eliminate ω from the expression for the frictional force. Since $\omega = (\mathbf{C} + (RM\mathbf{v} \times \ \mathbf{k})/I$, $I = \beta \ R^2 M$, and $\mathbf{v} \times \ \mathbf{k} \times \ \mathbf{k} = -\mathbf{v}$, we have

$$\begin{aligned} \mathbf{F} &= -\mu \ Mg < \mathbf{v} - R\omega \times \ \mathbf{k} > \\ &= -\mu \ Mg < \mathbf{v} - R/I(\mathbf{C} + RM\mathbf{v} \times \mathbf{k}) \times \ \mathbf{k} > \\ &= -\mu \ Mg < \mathbf{v} - R/I\mathbf{C} \times \ \mathbf{k} + R^2M/I\mathbf{v} > \\ &= -\mu \ Mg < (1 + \beta^{-1})\mathbf{v} - R/I\mathbf{C} \times \ \mathbf{k} > \\ &= -\mu \ Mg < \mathbf{v} - \frac{\mathbf{C} \times \ \mathbf{k}}{(1+\beta)RM} > . \end{aligned}$$

where we have multiplied by a positive constant without changing the direction of the unit vector.

We can immediately deduce the velocity that results after sufficient time has elapsed that the ball is rolling on the lane. At that time (denoted with subscript f), the frictional force is zero, $\mathbf{F}_f = 0$ and hence $\mathbf{v}_f = \mathbf{C} \times \ \mathbf{k}/((1+\beta)RM)$. Now use the expression for $\mathbf{C}$ and I,

$$\begin{aligned} \mathbf{v}_f &= (I\omega_0 - RM\mathbf{v}_0 \times \ \mathbf{k}) \times \ \mathbf{k}/((1+\beta)RM) \\ &= (I\omega_0 \times \ \mathbf{k} + RM\mathbf{v}_0)/((1+\beta)RM) \\ &= 1/(1+\beta)\mathbf{v}_0 + \beta \ /(1+\beta)R\omega_0 \times \ \mathbf{k}. \end{aligned}$$

What is amazing about this formula is that it says the final velocity, in both magnitude and direction, is independent of the mass of the ball and of the coefficient of friction, even though the later may vary along the lane. Note that the quantity $\mathbf{v}_e = R\omega_0 \times \mathbf{k}$ is the velocity that the ball would have if it were rolling without friction on the lane with angular velocity ω_0.

The final angular velocity can now be found from $\mathbf{C}$,

$$\begin{aligned}\omega_f &= \mathbf{C}/I + RM/I\mathbf{v}_f \times \mathbf{k} \\ &= \omega_0 - 1/\beta R\mathbf{v}_0 \times \mathbf{k} + 1/\beta R(1/(1+\beta)\mathbf{v}_0 + \beta/(1+\beta)R\omega_0 \times \mathbf{k}) \times \mathbf{k} \\ &= (-1/\beta + 1/\beta(1+\beta))R^{-1}\mathbf{v}_0 \times \mathbf{k} + \omega_0 + 1/(1+\beta)\omega_0 \times \mathbf{k} \times \mathbf{k} \\ &= 1/(1+\beta)(-R^{-1}\mathbf{v}_0 \times \mathbf{k}) + \beta/(1+\beta)\omega_0^{xy} + \omega_0^z.\end{aligned}$$

The notation ω_0^{xy} and ω_0^z is for the xy and z component of ω where we have used $\omega \times \mathbf{k} \times \mathbf{k} = -\omega^{xy}$. Note that the vertical component of the angular velocity is unchanged. If we define the quantity $\omega_e = -R^{-1}\mathbf{v}_0 \times \mathbf{k} + \omega_0^z$ this corresponds to the angular velocity of a ball rolling without friction on the lane at the speed $\mathbf{v}_0$ and having vertical component the same as ω_0.

Summarizing,

$$\begin{aligned}\mathbf{v}_f &= 1/(1+\beta)\mathbf{v}_0 + \beta/(1+\beta)\mathbf{v}_e \\ \omega_f &= \beta/(1+\beta)\omega_0 + 1/(1+\beta)\omega_e.\end{aligned}$$

It is now possible to completely solve the equations of motion for the rolling ball. We do this first assuming that the coefficient of friction is constant during the motion. It should be obvious now that $\mathbf{F} = -\mu Mg < \mathbf{v} - \mathbf{v}_f >$. Therefore Newton's equation becomes

$$M\frac{d\mathbf{v}}{dt} = -\mu Mg < \mathbf{v} - \mathbf{v}_f >$$

or

$$\frac{d(\mathbf{v} - \mathbf{v}_f)}{dt} = -\mu g < \mathbf{v} - \mathbf{v}_f > .$$

Note that this implies that the direction of $\mathbf{v} - \mathbf{v}_f$ is not changed during the motion. Therefore it should be easy to see that the solution is

$$\mathbf{v} - \mathbf{v}_f = (\mathbf{v}_0 - \mathbf{v}_f) - \mu g < \mathbf{v}_0 - \mathbf{v}_f > t$$

for $0 \le t \le |\mathbf{v}_0 - \mathbf{v}_f|/\mu g$, and $\mathbf{v} = \mathbf{v}_f$ afterwards. Alternatively, if $\alpha = \mu g < \mathbf{v}_0 - \mathbf{v}_f >$ then this can be written

$$\mathbf{v} = \mathbf{v}_0 - \alpha t$$

which is valid until $\mathbf{v} = \mathbf{v}_f$. From this we see that the trajectory follows a parabola over the section of the lane where μ is constant. If $\mathbf{r}(t)$ is the position of the ball at time t, then

$$\mathbf{r}(t) = \mathbf{r}_0 + \mathbf{v}_0 t - \frac{1}{2}\alpha t^2.$$

It should be pointed out now that although the final velocity $\mathbf{v}_f$ does not depend on the mass or the coefficient of friction, the trajectory is also independent of the mass but depends on the coefficient of friction. It is the precise trajectory that determines how the ball will hit the pins, and this is the bowler's object to control. Even assuming that lane conditions are maintained so that most balls will reach their final velocity before hitting the pins, the exact location of that final trajectory is not certain. Explicitly, consider the extended line on the $x-y$ plane that the final trajectory falls on. What is the equation of this line? Assume first that μ is constant on the lane. Then, the final trajectory is reached at $t_f = |\mathbf{v}_0 - \mathbf{v}_f|/\mu g$ so we can substitute into the expression for $\mathbf{r}$ to obtain

$$\begin{aligned}\mathbf{r}_f &= \mathbf{r}_0 + \mathbf{v}_0|\mathbf{v}_0 - \mathbf{v}_f|/\mu g - \frac{1}{2}(\mathbf{v}_0 - \mathbf{v}_f)|\mathbf{v}_0 - \mathbf{v}_f|/\mu g \\ &= \mathbf{r}_0 + \frac{|\mathbf{v}_0 - \mathbf{v}_f|}{\mu g}\frac{\mathbf{v}_0 + \mathbf{v}_f}{2}.\end{aligned}$$

We now have a point and known direction for this line, which we can take in parametric form to be

$$\mathbf{r}(s) = \mathbf{r}_f + \mathbf{v}_f s.$$

Where the expression for $\mathbf{r}_f$ shows the explicit dependence on μ.

Coefficient of Friction

The coefficient of friction of a bowling ball on a lane surface is determined by three factors: the surface of the ball, the surface of the lane, and the dressing or coating of oil on the lane. There are a variety of ball surfaces that may be employed, and a variety of lane surfaces. Suffice it to say that for a given ball and given lane, there is a well defined coefficient of friction, which we will call μ for the ball sliding on the bare lane. The unknowable factor is the type, quantity, and distribution of oil on the lane surface. In a common system of bowling, a coating of oil is applied over the first 30 to 45 feet of the lane closest to the bowler (a lane is 60 feet long), with the remaining portion of the lane unoiled. The effect of the oil, naturally, is to reduce, or even effectively eliminate, the friction. Where there is no friction, a ball will travel in a straight line and its angular velocity will not change. If the ball then reaches a portion of the lane at a position r_0 that is unoiled, having its initial velocity and angular velocity, the trajectory over the remaining lane is given by the parabola described above.

If the exact distribution of oil on the lane were known, it would theoretically be possible to solve the above equations for the exact trajectory of the ball. Unfortunately, this is not the condition that is observed. Oil evaporates over time, and a portion is scraped off and transported down the lane by successive balls. But all is not lost. The final velocity of the ball is predicted from the conservation of angular momentum, and this implies the *angle* at which the ball attacks the pins. And assuming the ideal distribution of oil will give a very good approximation to the actual trajectory.

Pin Action

With a ball trajectory now described based on initial conditions and assumptions on lane conditions, we wish to translate this into a prediction of which pins will be knocked down and which will remain standing. The 10 pins are arranged in a triangular grid of equilateral triangles spaced $12''$ apart. The pins themselves are about $4.8''$ in diameter at the height that a ball will strike a standing pin. At the same time, the diameter of a bowling ball is about $8.6''$. This means that a ball will not fit between adjacent pins without hitting one, and may hit both. When a pin is struck, it rebounds away from the ball and may go on to strike another pin, which may then rebound and hit another, etc. A pin may hit more than one pin this way, and may even fly off of the lane and rebound from a side wall back onto the lane to strike pins legally (the ball is not allowed to do this). It can be seen that many strikes (all 10 pins knocked down) depend on this side wall action.

When a ball hits a pin, it is possible to predict with reasonable simplicity the direction and trajectory that the pin will initially follow. However, because a pin is flying through the air and toppling, there are many different ways that it may come into contact with another pin, lane, or side wall, unlike the ball which has only one way of striking a standing pin. Consequently, the mathematical effort required to describe this accurately may be overwhelming.

Nevertheless, it is possible to accurately predict the direction and momentum of the ball and pin after a collision. With this we can predict accurately whether the pin will or will not strike another pin. We can also make the reasonable assumption that if a pin strikes another pin, that there is a high likelihood that the entire row of pins will be knocked down (if there are pins continuing the row). We will not attempt to predict any other kinds of secondary collisions. Our result therefore will give a high probability of pins that are certain to be knocked down, although in reality more pins may be knocked down. All that we require is the result of the collision of a rolling ball and standing pin.

Collision of Ball with Pins

Qualitatively, the center of mass of a bowling pin is slightly higher than the point of collision with a bowling ball. Furthermore, the point of impact is very slightly below the widest diameter of the pin, and therefore the pin is facing slightly towards the lane at this point. This suggests, and it is seen in video replays, that when a ball strikes a pin, the pin flies in a direction that takes it off of the lane. As it is flying on this trajectory, the bottom end of the pin travels slightly faster than the top end, causing the pin to lean towards the ball as it flies away. It is also seen that there is very little noticable spin imparted to the pin (contrary to the popular theory that the spin on the ball causes the pins to spin around when they are hit).

Similarly, the bowling ball is deflected, and we know that the spin of the ball will cause friction on the lane. However, we will ignore this effect because it takes place in such a short period of time that it is negligable.

When a pin then strikes a second pin directly, the effect is to reverse the direction it is leaning and slow it down. Glancing blows of two pins are much less predictable.

Having described all of this, we are only interested in the horizontal component of the direction that the pin is moving after a collision, and the direction that the ball is moving. The geometry of this collision is no different from the collision of two balls of slightly different size, though we will assume the two balls are the same size but of different mass. We now derive this relationship, which will be valid in many situations.

The physical principles required in this derivation are conservation of momentum and coefficient of restitution. Two balls colliding interact along the direction perpendicular to the impact. In an elastic collision, energy is conserved and this information is sufficient to derive the resulting velocities of the two balls. However few collisions are truly elastic, energy is lost in vibration and heat in the balls. It is emperically known that for any two objects colliding head on, the speed of separation after the collision is proportional to the speed of collision. This proportionality constant is the *coefficient of restitution*, c. For elastic collisions, $c = 1$, but for objects in the real world, $0 < c < 1$.

In a collision that is not head-on, the coefficient of restitution applies to the velocity perpindicular to the collision. For example, suppose a ball of mass M travels in the xy plane with a velocity $\mathbf{v}_i$ and collides with a wall on the x-axis. Then $v_f^y = -cv_i^y$ and $v_f^x = v_i^x$, or

$$\mathbf{v}_f = \begin{pmatrix} 1 & 0 \\ 0 & -c \end{pmatrix} \mathbf{v}_i.$$

A ball does not reflect like a beam of light when the collision is inelastic. In general, if the ball collides with a wall making an angle θ with the x-axis, then

$$\begin{aligned} \mathbf{v}_f &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -c \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \mathbf{v}_i \\ &= \begin{pmatrix} \cos^2\theta \ - c\sin^2\theta & (1+c)\sin\theta\ \cos\theta \\ (1+c)\sin\theta\ \cos\theta & \sin^2\theta \ - c\cos^2\theta \end{pmatrix} \mathbf{v}_i. \end{aligned}$$

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Now suppose there are two balls 1 and 2 with mass M and m, and radii R and r respectively. To begin with, assume the balls are moving initially with velocities $\mathbf{v}_i^1$ and $\mathbf{v}_i^2$ such that the center of mass is stationary, i.e. the total momentum is 0. Let us also assume that the point of collision is the origin and at the moment of collision the tangent at the point of collision makes an angle θ with the x-axis. Then the expression above gives the velocities after the collision in this frame

$$\mathbf{v}_c^b = \begin{pmatrix} \cos^2\theta \ - c\sin^2\theta & (1+c)\sin\theta\ \cos\theta \\ (1+c)\sin\theta\ \cos\theta & \sin^2\theta \ - c\cos^2\theta \end{pmatrix} \mathbf{v}_i^b.$$

for each of the balls $b = 1, 2$. It is easy to see that the total momentum is 0 after the collision since it was 0 before, i.e. momentum is conserved.

In our situation, ball 2 is not moving. So to get to the stationary coordinate system, we must add velocity $\mathbf{v}_i^2$ to both balls such that

$$M(\mathbf{v}_i + \mathbf{v}_i^2) + m\mathbf{v}_i^2 = 0.$$

That is, $\mathbf{v}_i^2 = -M/(M+m)\mathbf{v}_i$ and so $\mathbf{v}_i^1 = m/(M+m)\mathbf{v}_i$. To get back to the coordinate system of the lane, $\mathbf{v}_f^b = \mathbf{v}_c^b - \mathbf{v}_i^1$.

All that remains is to determine the angle θ and this depends on the actual trajectories of the rolling ball. In fact, the angle of the impact plane is the same in both coordinate systems.

To be continued...

Rotation and Collision

Although the effect of rotation is minimal in the collision of the bowling ball with pins, the above equations apply equally well to billiard balls where the coefficient of friction is very high, and rotation is very important. What is more, the friction action between two balls, and the friction between the ball and bumper at the point of impact is an important factor. We illustrate for the rebound of a rolling billiard ball from a bumper on the x-axis. The ball has an initial velocity given by $\mathbf{v}_i$. We can assume that the vertical component of the angular velocity is zero, since it is unchanged throughout the motion anyway. So we take the initial rotation to be $\omega_i = (-R^{-1})\mathbf{v}_i \times \ \mathbf{k}$ or

$$\omega_i = R^{-1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{v}_i.$$

From the definition given above, $\mathbf{v}_e = R\omega_i \times \ \mathbf{k} = \mathbf{v}_i$.

We now give a careful derivation of the effect of friction at the point of impact. Suppose the trajectory of the ball is given by a force $F(t)\mathbf{j}$ depending on a time (during the short duration of the impact). From the direction of the ball's spin, the frictional force is in the $\mathbf{k}$ direction and it is proportional to the normal force, the coefficient of friction will be denoted ν. Thus the force of friction on the ball during the impact is $\nu \ F(t)\mathbf{k}$ (we assume this force is not great enough to lift the ball off the table). This generates a torque on the center of mass of the ball equal to $-R\nu \ F(t)\mathbf{i}$.

Newton's laws during the collision are

$$\begin{aligned} M\frac{d\mathbf{v}}{dt} &= F(t)\mathbf{j} \\ I\frac{d\omega}{dt} &= -R\nu \ F(t)\mathbf{i}. \end{aligned}$$

Now integrate both sides of these equations over the short time interval of the collision to get

$$\begin{aligned} M(\mathbf{v}_b - \mathbf{v}_i) &= J\mathbf{j} \\ I(\omega_b - \omega_i) &= -R\nu \ J\mathbf{i}. \end{aligned}$$

Where J is the integral of $F(t)$ over the interval, and is known as the impulse. The subscript b denotes the values immediately after the impact. Eliminating J we obtain

$$I(\omega_b - \omega_i) = -R\nu \ M(\mathbf{v}_b - \mathbf{v}_i) \times \ \mathbf{k}.$$

Now write out the cross products and use the expression for $\mathbf{v}_b$ and ω_i assuming a coefficient of restitution c,

$$\begin{aligned} I(\omega_b - \omega_i) &= -R\nu \ M \begin{pmatrix} 0 & -c-1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{v}_i \\ \omega_b &= R^{-1} \begin{pmatrix} 0 & -1 + \nu \ /\beta \ (1+c) \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{v}_i. \end{aligned}$$

The values of $\mathbf{v}_b$ and ω_b give the conditions immediately after the impact. We want to find the final velocity of the ball taking surface friction into account. The value of $\mathbf{v}_e$ is $R\omega_b \times \ \mathbf{k}$ or

$$\mathbf{v}_e = \begin{pmatrix} 1 & 0 \\ 0 & -1 + \nu \ /\beta \ (1+c) \end{pmatrix} \mathbf{v}_i.$$

Consequently the final velocity is given by

$$\begin{aligned} \mathbf{v}_f &= 1/(1+\beta)\mathbf{v}_b + \beta \ /(1+\beta)\mathbf{v}_e \\ &= \begin{pmatrix} 1 & 0 \\ 0 & (-c + \beta \ + \nu \ (1+c))/(1+\beta) \end{pmatrix} \mathbf{v}_i. \end{aligned}$$

This is equivalent to a collision with coefficient of restitution equal to

$$c_{eff} = \frac{c - \beta \ - \nu \ (1 + c)}{1 + \beta}.$$

Even for a perfectly elastic collision ($c = 1$) with no friction on a bumper ($\nu = 0$), the equivalent coefficient of restitution is $(1 - \beta)/(1 + \beta)$ which is only 3/7 for a uniform sphere. The actual trajectory after the collision is not a straight line, however, but a parabola as described above. With high coefficient of friction, the trajectory becomes a straight line quickly. A pool player needs only to learn how to predict a constant coefficient of restitution c_{eff} which is nearly the same for virtually all pool tables, and can shoot accurately with no knowledge of the effect of the rotation on the collision. A similar result no doubt will be found for a collision of two balls, where the coefficient of friction and restitution, and the resulting c_{eff} will differ from the value on the bumper. Again, the effect of rotation on the ball can be ignored.

Rotation and Rolling

The results derived concerning balls rolling and sliding on surfaces and rebounding from other objects is completely general. To complete the discussion of another related sport, we consider the problem of the trajectory of a golf ball on a putting green. In this case, the coefficient of friction is very high and it may be assumed that the ball is rolling perfectly on the surface at all times. But the surface is not flat, and this gives rise to down-hill forces.

Frictional force $\mathbf{F}_f$ acts on the ball at the point of contact with the ground and it satisfies Newton's equation,

$$I\frac{d\omega}{dt} = (-R\mathbf{k}) \times \ \mathbf{F}_f.$$

Since the force is planar, $\mathbf{F}_f = -(\mathbf{F}_f \times \ \mathbf{k}) \times \ \mathbf{k}$, so

$$\mathbf{F}_f = -R^{-1} I \frac{d\omega}{dt} \times \ \mathbf{k}.$$

Now since the ball is always rolling, $\mathbf{v} = R\omega \times \ \mathbf{k}$, so

$$\frac{d\mathbf{v}}{dt} = R\frac{d\omega}{dt} \times \ \mathbf{k}$$

or

$$\mathbf{F}_{app} = -R^{-2} I \frac{d\mathbf{v}}{dt}.$$

Now we can say that if there is a general force $\mathbf{F}$ acting on the ball, then

$$M\frac{d\mathbf{v}}{dt} = \mathbf{F} - R^{-2} I \frac{d\mathbf{v}}{dt}.$$

Letting $I = \beta \ R^2 M$,

$$M(1 + \beta)\frac{d\mathbf{v}}{dt} = \mathbf{F}.$$

The effect of rotation is to make the ball respond to forces as if it is $1 + \beta$ times its actual mass. For example $\beta = 2/5$, we see that the ball acts as if it were 40% heavier in Newton's equation.

The force of a ball on a putting green has two components: gravitation and drag. If the height relative to a fixed reference height of the green at a point (x, y) is given by $h(x, y)$, then the potential energy of the ball when it is at this point is given by $Mgh(x, y)$. Therefore, the force on the ball due to gravity on the green is given by $-Mg\nabla h(x, y)$. The other force is the force of drag discussed in the beginning. This force also acts on the ball at the point of contact (and takes part in keeping the ball rolling), but it naturally has an effect on the center of mass. The drag force for ball velocities seen on a putting green can probably be approximated by a constant force in a direction opposite to the direction of motion. The magnitude of this force should be determined emperically, and the dependence on the velocity should also be confirmed.

Simulation of Pin Action

To be continued...